Differential Geometry II - Smooth Manifolds
Winter Term 2025/2026
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Exercise Sheet 7 – Solutions

Exercise 1:

(a) *Sufficient conditions for properness*: Let X and Y be topological spaces and let $F\colon X \to Y$ be a continuous map. Prove the following assertions:

   (i) If X is compact and Y is Hausdorff, then F is proper.

   (ii) If F is a topological embedding with closed image, then F is proper.

   (iii) If Y is Hausdorff and F has a continuous *left inverse*, i.e., a continuous map $G\colon Y \to X$ such that $G \circ F = \mathrm{Id}_X$, then F is proper.

(b) Let M be a smooth manifold and let S be an embedded submanifold of M. Show that S is properly embedded if and only if S is a closed subset of M.

(c) *Global graphs are properly embedded*: Let $f\colon M \to N$ be a smooth map between smooth manifolds. Show that the graph $\Gamma(f)$ of f is a properly embedded submanifold of $M \times N$.

Solution:

(a) We deal with the three cases below separately:[1]

   (i) Let K be a compact subset of Y. Since Y is Hausdorff, K is a closed subset of Y. Since F is continuous, $F^{-1}(K)$ is a closed subset of X, and since X is compact, $F^{-1}(K)$ is also compact, as desired.

   (ii) Let K be a compact subset of Y. By assumption, $F(X)$ is a closed subset of X, so $F(X) \cap K$ is a closed subset of K, and thus compact. Since $F^{-1}\colon F(X) \to X$ is continuous and bijective by assumption and since $F(X) \cap K \subseteq F(X)$, the image $F^{-1}\big(F(X) \cap K\big) = F^{-1}(K)$ is a compact subset of X, as desired.

[1] Recall that a (continuous) map $F\colon X \to Y$ between topological spaces is said to be *proper* if for every compact subset K of Y, the preimage $F^{-1}(K)$ is a compact subset of X.

(iii) Let K be a compact subset of Y. On the one hand, since G is continuous, $G(K)$ is a compact subset of X. On the other hand, since Y is Hausdorff, K is a closed subset of Y, and since F is continuous, $F^{-1}(K)$ is a closed subset of X. Now, we claim that $F^{-1}(K) \subseteq G(K)$, which implies that $F^{-1}(K)$ is compact, as desired. Indeed, given $s \in F^{-1}(K)$, we have $F(s) = t \in K$, so

$$s = \mathrm{Id}_X(s) = (G \circ F)(s) = G(t) \in G(K),$$

which proves the claim, and completes thus the proof of (iii).

(b) Assume first that S is a properly embedded submanifold of M. Then the inclusion map $\iota\colon S \hookrightarrow M$ is proper by definition, and hence closed by *Claim 3* in the proof of *Proposition 4.6*. Since ι is clearly a topological embedding, we deduce that S is a closed subset of M.

Assume now that S is a closed subset of M. Since the inclusion map $\iota\colon S \hookrightarrow M$ is then a topological embedding with closed image $\iota(S) = S$, it follows from (a)(ii) that ι is proper, and thus S is a properly embedded submanifold of M.

(c) By *Example 5.5* we know that the map

$$\gamma_f\colon M \to M \times N,\ x \mapsto \big(x, f(x)\big)$$

is a smooth embedding with image $\Gamma(f)$ and that the projection

$$\pi_M\colon M \times N \to M,\ (x,y) \mapsto x$$

is a smooth left inverse for γ_f, i.e., $\pi_M \circ \gamma_f = \mathrm{Id}_M$. It follows from (a)(iii) that γ_f is proper, hence closed by *Claim 3* in the proof of *Proposition 4.6*. Therefore, $\Gamma(f)$ is a closed subset of $M \times N$, so *Example 5.5* together with (b) imply that $\Gamma(f)$ is a properly embedded submanifold of $M \times N$.

Exercise 2: Fix $n \geq 0$. Using

(i) the local slice criterion, and

(ii) the regular level set theorem,

show that $\mathbb{S}^n$ is an embedded submanifold of $\mathbb{R}^{n+1}$.

Solution:

(i) Recall that $\mathbb{S}^n$ is locally the graph of a smooth function. Indeed, by *Example 1.3*(2) we already know that each point of $\mathbb{S}^n$ belongs to one of the sets $U_i^{\pm} \cap \mathbb{S}^n$, where $U_i^+ \cap \mathbb{S}^n$ is the graph of

$$x^i = f\big(x^1, \dots, \widehat{x^i}, \dots, x^{n+1}\big)$$

and $U_i^- \cap \mathbb{S}^n$ is the graph of

$$x^i = -f\big(x^1, \dots, \widehat{x^i}, \dots, x^{n+1}\big),$$

where f is the smooth function

$$f\colon \mathbb{B}^n \to \mathbb{R},\ u \mapsto \sqrt{1-\|u\|^2}.$$

It follows now from *Example 5.5* and *Theorem 5.9* that $\mathbb{S}^n$ satisfies the local n-slice criterion, and hence it is an embedded submanifold of $\mathbb{R}^{n+1}$ again by *Theorem 5.9.*

(ii) Consider the smooth function

$$f\colon \mathbb{R}^{n+1} \to \mathbb{R},\ x=(x^1,\dots,x^{n+1}) \mapsto \|x\|^2-1=\sum_{i=1}^{n+1}(x^i)^2-1$$

and note that

$$\mathbb{S}^n = f^{-1}(0).$$

The gradient of f at an arbitrary point $x=(x^1,\dots,x^{n+1}) \in \mathbb{R}^{n+1}$ is given by

$$\operatorname{grad}(f)(x^1,\dots,x^{n+1}) = \big(2x^1,\dots,2x^{n+1}\big).$$

Since $\operatorname{grad}(f)$ vanishes only at the point $0=(0,\dots,0)\in\mathbb{R}^{n+1}$, which clearly does not belong to $\mathbb{S}^n$, it follows from *Corollary 5.13* that $\mathbb{S}^n = f^{-1}(0)$ is a properly embedded submanifold of $\mathbb{R}^{n+1}$.

Remark.

(1) It follows from *Exercise* 1(b) and *Exercise* 2, or [*Exercise Sheet* 5*, Exercise* 4(*c*)] and the compactness of $\mathbb{S}^n$, that $\mathbb{S}^n$ is a properly embedded submanifold of $\mathbb{R}^{n+1}$.

(2) One can check that the coordinates for $\mathbb{S}^n$ determined by the slice charts described in *Exercise* 2(i) are precisely the graph coordinates defined in *Example 1.3*(2).

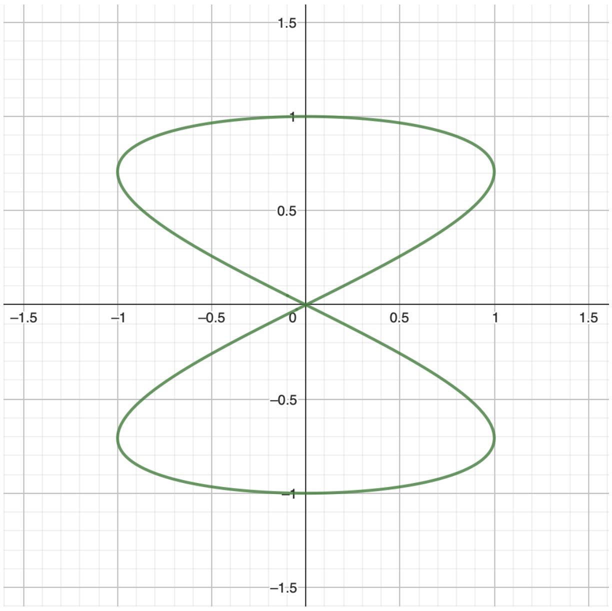
Exercise 3: Consider the smooth curve

$$\beta\colon (-\pi,\pi) \to \mathbb{R}^2,\ t \mapsto (\sin 2t,\, \sin t)$$

from *Example 4.5*(2). Show that its image is not an embedded submanifold of $\mathbb{R}^2$.

[Be careful: this is not the same as showing that β is not a smooth embedding.]

Solution: Endowed with the subspace topology inherited from $\mathbb{R}^2$, the image of β (which has been plotted below) is not a topological manifold. Indeed, essentially the same argument as the one presented in the solution of [*Exercise Sheet* 1*, Exercise* 4] shows that $\beta(-\pi,\pi)$ is not locally Euclidean at the (self-intersection) point $(0,0)\in\beta(-\pi,\pi)$. Thus, the image of β cannot be an embedded submanifold of $\mathbb{R}^2$.



Exercise 4 (to be submitted):

(a) Consider the map

$$F\colon \mathbb{R}^4 \to \mathbb{R}^2,\ (x, y, s, t) \mapsto (x^2 + y,\ x^2 + y^2 + s^2 + t^2 + y).$$

Show that $(0, 1) \in \mathbb{R}^2$ is a regular value of F, and that the level set $F^{-1}(0, 1)$ is diffeomorphic to $\mathbb{S}^2$.

(b) Consider the smooth function

$$\Phi\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x^2 - y^2.$$

Given $c \in \mathbb{R}$, examine whether the corresponding level set $\Phi^{-1}(c)$ is an embedded submanifold of $\mathbb{R}^2$.

(c) Determine the regular values of the smooth function

$$\Psi\colon \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x^2 - y^3$$

and draw the level sets corresponding to its critical values.

Solution:

(a) Let $p = (x, y, s, t) \in F^{-1}(0, 1)$ be arbitrary. The Jacobian of F at p is given by

$$dF_p = \begin{pmatrix} 2x & 1 & 0 & 0 \\ 2x & 2y+1 & 2s & 2t \end{pmatrix}. \tag{1}$$

Thus, if either $s \neq 0$ or $t \neq 0$, then dF_p is surjective. On the other hand, if $s = t = 0$, then we have $x^2 + y = 0$ and $x^2 + y^2 + y = 1$, which implies $y = -1$ and $x = \pm 1$, so we have

$$dF_p = \begin{pmatrix} \pm 2 & 1 & 0 & 0 \\ \pm 2 & -1 & 0 & 0 \end{pmatrix}$$

in this case, which is also surjective. Hence, $(0,1)$ is a regular value of F.

Set $S := F^{-1}(0,1)$. It is a properly embedded submanifold of $\mathbb{R}^4$ by the above and *Corollary 5.13*. To show that S is diffeomorphic to $\mathbb{S}^2$, observe that by combining the equations $x^2+y=0$ and $x^2+y^2+s^2+t^2+y=1$, we obtain $y^2+s^2+t^2=1$. However, the map $(x,y,s,t)\mapsto(y,s,t)$ does not induce a diffeomorphism of S with $\mathbb{S}^2$, because it only covers the hemisphere where $y\leq 0$, as $y=-x^2$. Note however that by replacing y with $-x^2$, we also obtain the equation $x^4+s^2+t^2=1$, and through this equation we are going to construct a diffeomorphism.

Consider the set

$$U := \mathbb{R}^4 \setminus \{x=s=t=0\}$$

and the map

$$\begin{aligned} g\colon U &\to \mathbb{R}^3 \\ (x,y,s,t) &\mapsto \frac{1}{\sqrt{x^2+s^2+t^2}}(x,s,t). \end{aligned}$$

Note that the function $(x,s,t)\in\mathbb{R}^3\setminus\{0\}\mapsto(x^2+s^2+t^2)^{-1/2}\in\mathbb{R}_{>0}$ is smooth, since it is the composition of the smooth functions $(x,s,t)\in\mathbb{R}^3\setminus\{0\}\mapsto x^2+s^2+t^2\in\mathbb{R}_{>0}$ and $v\in\mathbb{R}_{>0}\mapsto v^{-1/2}\in\mathbb{R}_{>0}$. Hence, g is smooth as well. Note that $S\subseteq U$, and thus the restriction $g|_S\colon S\to\mathbb{R}^3$ is also smooth by *Exercise* 5(a). As its image is contained in $\mathbb{S}^2$ and since this is an embedded submanifold of $\mathbb{R}^3$ by *Exercise* 2, it follows from *Exercise* 5(c) that the corestriction $h := g|_S^{\mathbb{S}^2}$ is smooth. We will show that h is a diffeomorphism.

First, to check that h is surjective, let $(x',s',t')\in\mathbb{S}^2$ be arbitrary. Using that

$$(x')^2+(s')^2+(t')^2=1,$$

one may readily verify that the equation

$$(\lambda x')^4+(\lambda s')^2+(\lambda t')^2=1$$

admits a solution $\lambda\in\mathbb{R}\setminus\{0\}$. We then let $(x,y,s,t)=(\lambda x',-(\lambda x')^2,\lambda s',\lambda t')$, and it is then straightforward to verify that $(x,y,s,t)\in S$ and $g(x,y,s,t)=(x',s',t')$, so h is surjective.

Next, to see that h is injective, suppose that $(x,y,s,t),(x',y',s',t')\in S$ are such that $h(x,y,s,t)=h(x',y',s',t')$. This means that there exists $\lambda>0$ such that $\lambda(x,s,t)=(x',s',t')$. But then we also have

$$x^4+s^2+t^2=1 \quad\text{and}\quad (\lambda x)^4+(\lambda s)^2+(\lambda t)^2=1.$$

Since the function $z\in\mathbb{R}_{>0}\mapsto x^4z^4+(s^2+t^2)z^2$ is strictly increasing, we deduce that $\lambda=1$, which then implies $(x,y,s,t)=(x',y',s',t')$, as desired.

We conclude that $h\colon S\to\mathbb{S}^2$ is bijective, so it remains to see that its differential is everywhere invertible; see *Proposition 4.9*(f). Since both S and $\mathbb{S}^2$ are two-dimensional, it suffices to see that the differential is everywhere injective. Let $p=(x,y,s,t)\in S$ be arbitrary. As a subspace of $T_p\mathbb{R}^4$, the tangent space T_pS is given by the kernel of dF_p thanks to *Proposition 5.23*. Therefore, to conclude that dh_p is injective, we have to show that $\ker(dF_p)$ and $\ker(dg_p)$ have trivial intersection.

We start by computing $\ker(dg_p)$. Note that g remains constant under changing the y–coordinate of p, and under scaling p. This means that $(0,1,0,0)$, $p \in \ker(dg_p)$ under the usual identifications. In what follows we will show that these two vectors generate $\ker(dg_p)$ (note that they are linearly independent as $p \in U$). To this end, we use a trick similar to the one appearing in the solution to [*Exercise Sheet* 5*, Exercise* 2(*c*)]; namely, we construct a section of $g|^{\mathbb{S}^2}$ – of course, we could also just compute the Jacobian of g, but this would simply be a lot of work. Denote by $j\colon \mathbb{S}^2 \to \mathbb{R}^3$ the natural inclusion, so that $g = j \circ g|^{\mathbb{S}^2}$, and consider the map

$$\begin{aligned}\sigma\colon \mathbb{S}^2 &\to U\\ (x', s', t') &\mapsto (rx', y, rs', rt'),\end{aligned}$$

where $r = \sqrt{x^2+s^2+t^2}$. This is clearly smooth, and it is straightforward to check that $g|^{\mathbb{S}^2} \circ \sigma = \mathrm{Id}_{\mathbb{S}^2}$. Hence, we have

$$g \circ \sigma = j \circ g|^{\mathbb{S}^2} \circ \sigma = j.$$

So if we set $q = (x/r,\ s/r,\ t/r) \in \mathbb{S}^2$, then $\sigma(q) = p$ and thus

$$dg_p \circ d\sigma_q = dj_q\,.$$

As dj_q is injective, it follows that $d\sigma_q$ is injective, and that $\ker(dg_p)$ and $\mathrm{im}(d\sigma_q)$ intersect trivially. As hence the image of $d\sigma_q$ is two-dimensional, we infer that the dimension of $\ker(dg_p)$ can be at most $2 = 4 - 2$. Consequently,

$$\ker(dg_p) = \mathrm{span}\big(p,\ (0,1,0,0)\big).$$

Now let us finish the exercise by proving that $\ker(dg_p)$ and $\ker(dF_p)$ have trivial intersection. Let $(a,b,c,d) \in \ker(dg_p)$. By the above description we deduce that

$$\exists \lambda \in \mathbb{R} \text{ such that } (a,c,d) = \lambda(x,s,t). \tag{2}$$

If we also assume that $(a,b,c,d) \in \ker(dF_p)$, then by (1) and (2) we obtain the following system:

$$\begin{aligned}2x^2\lambda + b &= 0\\ 2x^2\lambda + (2y+1)b + 2s^2\lambda + 2t^2\lambda &= 0.\end{aligned}$$

Combining the two equations gives

$$\lambda\big(2x^2 - (2y+1)2x^2 + 2s^2 + 2t^2\big) = 0.$$

Using that $y = -x^2$ and after simplyfing, we deduce that the paranthesis is equal to $2(x^4 + s^2 + t^2) = 2$, and thus $\lambda = 0$. Therefore, $(a,b,c,d) = 0$, and thus $\ker(dF_p)$ and $\ker(dg_p)$ have trivial intersection, as desired.

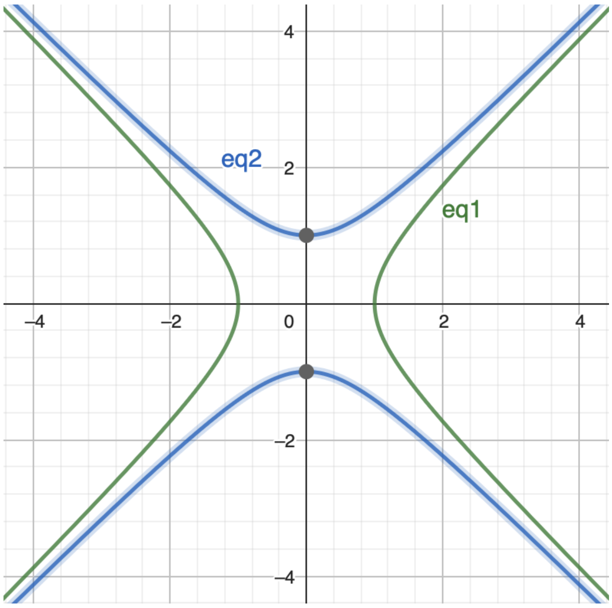
(b) The gradient of Φ at an arbitrary point $(x,y) \in \mathbb{R}^2$ is given by

$$\mathrm{grad}(\Phi)(x,y) = \left(\frac{\partial \Phi}{\partial x}(x,y),\ \frac{\partial \Phi}{\partial y}(x,y)\right) = \big(2x, -2y\big)$$

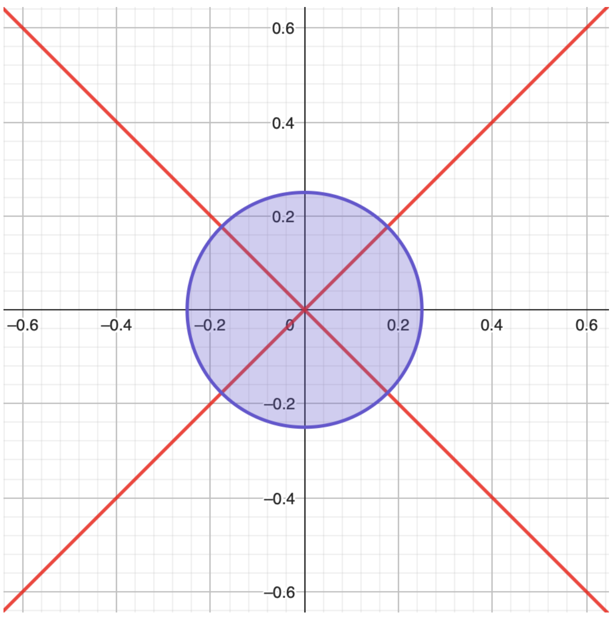
and it is obvious that

$$\mathrm{grad}(\Phi)(x, y) = (0, 0) \ \text{ if and only if } \ (x, y) = (0, 0).$$

Since any two fibers of Φ are disjoint and since we clearly have $\Phi(0, 0) = 0$, we conclude that if $c \neq 0$, then the corresponding level set $\Phi^{-1}(c)$ is a properly embedded submanifold of $\mathbb{R}^2$ by *Corollary 5.13*. – We have plotted below the level sets $\Phi^{-1}(1)$ (in green) and $\Phi^{-1}(-1)$ (in blue).



We now deal with the remaining case $c = 0$. Since $\mathrm{grad}(\Phi)(0, 0) = (0, 0)$, $c = 0$ is a critical value of Φ, so *Corollary 5.13* cannot be applied; we stress that it does *not* tell us that the level set $\Phi^{-1}(0)$ is not an embedded submanifold of $\mathbb{R}^2$ either. To examine whether this is true or not, we plot below (in red) the level set $\Phi^{-1}(0)$ and we proceed as follows.



We observe that $\Phi^{-1}(0)$ is the union of the lines $y = x$ and $y = -x$ in the plane $\mathbb{R}^2$. By arguing as in [*Exercise Sheet* 1, *Exercise* 4] (for the point $(0,0) \in \Phi^{-1}(0)$), we infer that $\Phi^{-1}(0)$ is not a topological manifold with the subspace topology inherited from $\mathbb{R}^2$, and hence it cannot be an embedded submanifold of $\mathbb{R}^2$.

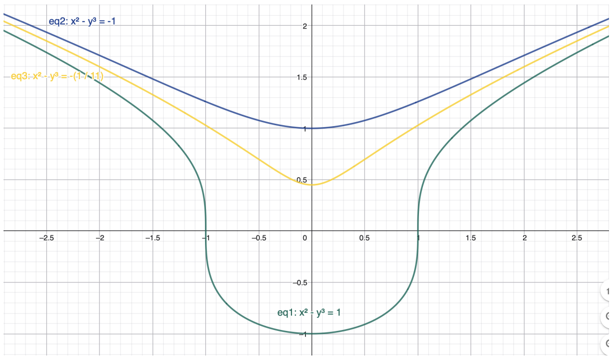
(c) The gradient of Ψ at an arbitrary point $(x,y) \in \mathbb{R}^2$ is given by

$$\mathrm{grad}(\Psi)(x,y) = \left(\frac{\partial\Psi}{\partial x}(x,y),\ \frac{\partial\Psi}{\partial y}(x,y)\right) = \left(2x, -3y^2\right)$$

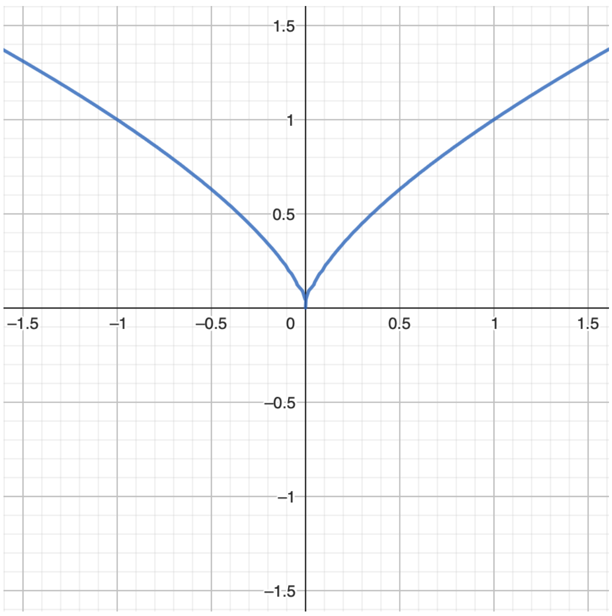
and it is obvious that

$$\mathrm{grad}(\Psi)(x,y) = (0,0) \ \text{ if and only if } \ (x,y) = (0,0).$$

Since any two fibers of Ψ are disjoint and since we clearly have $\Psi(0,0) = 0$, we conclude that any $c \in \mathbb{R} \setminus \{0\}$ is a regular value of Ψ, and hence the associated regular level set $\Psi^{-1}(c)$ is a properly embedded submanifold of $\mathbb{R}^2$ by *Corollary 5.13.* – We have plotted below the regular level sets $\Psi^{-1}(1)$ (in green), $\Psi^{-1}(-1)$ (in blue) and $\Psi^{-1}(-1/11)$ (in yellow)



as well as the critical level set $\Psi^{-1}(0)$ (in blue).



Exercise 5:

(a) *Restricting the domain of a smooth map*: If $F\colon M \to N$ is a smooth map and if $S \subseteq M$ is an immersed or embedded submanifold, then the restriction $F|_S\colon S \to N$ is smooth.

(b) *Restricting the codomain of a smooth map*: Let M be a smooth manifold, let $S \subseteq M$ be an immersed submanifold, and let $G\colon N \to M$ be a smooth map whose image is contained in S. If G is a continuous map from N to S, then $G\colon N \to S$ is smooth.

(c) Let M be a smooth manifold and let $S \subseteq M$ be an embedded submanifold. Then every smooth map $G\colon N \to M$ whose image is contained in S is also smooth as a map from N to S.

Solution:

(a) The inclusion map $\iota\colon S \to M$ is smooth for both immersed and embedded submanifolds. Hence, the restriction $F|_S = F \circ \iota$ is smooth as well.

(b) Let $p \in M$ and set $q = G(p) \in M$. To prove the smoothness of the corestriction $G|^S\colon N \to S$, we need to find charts of N and S containing p and q, respectively, such that the corresponding coordinate representation of $G|^S$ is smooth. As immersed submanifolds are locally embedded by *Proposition 5.20*, there exists a neighborhood V of q in S such that $\iota_V\colon V \hookrightarrow M$ is a smooth embedding. Thus, there exists a smooth chart (W,ψ) of M containing q which is a slice chart for V (note that it could very well be that $W \cap V \subsetneq W \cap S$, i.e., (W,ψ) might not be a slice chart for S). The fact that (W,ψ) is a slice chart means that $\big(V_0,\widetilde{\psi}\big)$ is a smooth chart for V, where $V_0 = V \cap W$ and $\widetilde{\psi} = \pi \circ \psi$, with $\pi\colon \mathbb{R}^n \to \mathbb{R}^k$ the projection onto the first $k = \dim S$ coordinates. Since $V_0 = \iota_V^{-1}(W)$ is open in V by continuity of ι_V, it is open in S in its given topology. Hence, $\big(V_0,\widetilde{\psi}\big)$ is also a smooth chart for S. Set $U := G^{-1}(V_0)$ and note that U is an open subset of N containing p (this is where we use the hypothesis that G is continuous into S). Choose a smooth chart (U_0,φ) for N such that $p \in U_0 \subseteq U$. Then the coordinate representation of the corestriction $G|^S\colon N \to S$ with respect to the charts (U_0,φ) and $\big(V_0,\widetilde{\psi}\big)$ is

$$\widetilde{\psi} \circ G|^S \circ \varphi^{-1} = \pi \circ \psi \circ G \circ \varphi^{-1},$$

which is smooth, because $G\colon N \to M$ is smooth by assumption. This proves the assertion.

(c) According to (b), we only have to show that the corestriction of any smooth map $G\colon N \to M$ to S remains continuous. This is derived immediately from the following general topological fact: *if $f\colon X \to Y$ is a continuous map between topological spaces X and Y, and if $B \subseteq Y$ and $A \subseteq X$ are arbitrary subsets endowed with the subspace topology, and such that $f(A) \subseteq B$, then $f|_A^B\colon A \to B$ is continuous.*

Let us verify the above result for the sake of completeness. Let $V \subseteq B$ be an open subset of B. By definition of the subspace topology, there exists an open subset $V' \subseteq Y$ such that $V' \cap B = V$. Hence,

$$\big(f|_A^B\big)^{-1}(V) = f^{-1}(V') \cap A,$$

which is open in A, since $f^{-1}(V')$ is open in X by continuity of f and since A is endowed with the subspace topology. Therefore, $f|_A^B$ is continuous.

Remark.

(1) Let $F\colon M \to N$ be a smooth map. *Exercise* 5(a) asserts that if the domain of F is restricted to a smooth submanifold S of M, then the restriction of F to S remains smooth. However, if the codomain of F is restricted, then the resulting map need not be smooth in general, as the following example shows, but *Exercise* 5(b) demonstrates that the failure of continuity is the only thing that can go wrong.

Consider the smooth map

$$\beta\colon (-\pi, \pi) \to \mathbb{R}^2,\ t \mapsto (\sin 2t,\, \sin t),$$

which is an injective smooth immersion; see *Example 4.5*(2). According to *Proposition 5.16*, its image $S \coloneqq \beta(-\pi, \pi)$ has a unique topology and smooth structure such that S is an immersed submanifold of $\mathbb{R}^2$ and such that β is a diffeomorphism onto its image S.

Consider now the smooth map

$$B\colon \mathbb{R} \to \mathbb{R}^2,\ t \mapsto (\sin 2t,\, \sin t)$$

and note that its image lies in S. As a map from $\mathbb{R}$ to S, B is not continuous, because $\beta^{-1} \circ B$ is not continuous at $t = \pi$.

(2) If M is a smooth manifold and if S is an immersed submanifold of M, then S is said to be *weakly embedded in* M if every smooth map $F\colon N \to M$ whose image lies in S is a smooth map as a map from M to S. *Exercise* 5(c) shows that embedded submanifolds are weakly embedded, while the previous example demonstrates that there are immersed submanifolds which are not weakly embedded.